Composition operators, Aleksandrov measures and value distribution of analytic maps in the unit disc

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Academic dissertation

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List of contributed articles

This doctoral dissertation consists of an introductory part and the following three journal articles:

[A] Pekka J. Nieminen and Eero Saksman:
Boundary correspondence of Nevanlinna counting functions for self-maps of the unit disc
Transactions of the American Mathematical Society **356** (2004), no. 8, pp. 3167–3187

[B] Pekka J. Nieminen and Eero Saksman:
On compactness of the difference of composition operators
Journal of Mathematical Analysis and Applications **298** (2004), no. 2, pp. 501–522

[C] Pekka J. Nieminen:
Compact differences of composition operators on Bloch and Lipschitz spaces
Submitted; preprint version published in Reports in Mathematics **436** (2006), Department of Mathematics and Statistics, University of Helsinki

Introductory part

Contents

1 Background and overview

The subject of this dissertation lies at the interface of analytic function theory and operator theory. Our principal topics are analytic composition operators, Aleksandrov measures and their interaction.

Let $\mathbb{C}$ denote the complex plane and $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ the open unit disc. If $\varphi\colon \mathbb{D} \to \mathbb{D}$ is an analytic map, the *composition operator* induced by φ is the linear operator C_φ that takes any analytic or harmonic function $f\colon \mathbb{D} \to \mathbb{C}$ to $f \circ \varphi$. That is,

$$C_\varphi f(z) = f(\varphi(z)), \qquad z \in \mathbb{D}.$$

The map φ is then called the *symbol* of the operator. A systematic study of composition operators began in the late 1960s and it has subsequently evolved into a massive amount of research literature. The general idea in this endeavour has been to relate the function-theoretic properties of φ to the behaviour of C_φ. Typically one restricts C_φ to a given Banach space of analytic or harmonic functions and seeks to characterize its operator-theoretic properties such as boundedness, compactness and spectra. One might also be interested in describing the structure of the whole set of composition operators acting on a given function space. As general references in the field we mention the monographs [8, 36].

Each analytic map $\varphi\colon \mathbb{D} \to \mathbb{D}$ also determines a family of positive Borel measures τ_α ($\alpha \in \partial\mathbb{D}$), defined on the unit circle $\partial\mathbb{D}$ by means of the Poisson representation

$$\operatorname{Re} \frac{\alpha + \varphi(z)}{\alpha - \varphi(z)} = \int_{\partial\mathbb{D}} \frac{1 - |z|^2}{|\zeta - z|^2}\, d\tau_\alpha(\zeta).$$

These measures are called the *Aleksandrov* (or *Clark*) *measures* of φ. Starting from the 1970s, they have found application in a variety of topics in function theory and operator theory. For instance, they are intimately connected to the boundary behaviour of the map φ. They have also proven very useful in the study of composition operators. The recent lecture notes [29] and survey articles [19,26] contain a wealth of information on the many roles of Aleksandrov measures.

We will now give a brief overview of the research carried out in this dissertation. In Article [A] we analyse the function-theoretic nature of Aleksandrov measures. We show that the Aleksandrov measures associated with a map φ can be obtained as boundary limits of a suitably refined version of the Nevanlinna counting function of φ. Thus these measures have a subtle connection to the value distribution theory of analytic maps in the unit disc. An incentive for our study comes from the theory of composition operators, where the counting function and Aleksandrov measures have both been used to characterize compact composition operators on the classical Hardy spaces.

In Articles [B] and [C] we study the question when the difference of two composition operators is compact or weakly compact on a given function space. This is motivated by the desire to understand the topological structure of the set of composition operators. In [B] we work in the classical setting of Hardy spaces and their relatives, and as a main tool we use Aleksandrov measures. In [C] we turn our attention to the standard Bloch space and the spaces of Lipschitz-continuous analytic functions on $\mathbb{D}$. The methods required here will be quite different, with hyperbolic and other non-Euclidean distances and derivatives in the disc playing an important role.

The rest of this introductory part is organized as follows. Sections 2 to 4 contain some preliminary material on composition operators, Nevanlinna counting functions and Aleksandrov measures. Instead of giving any kind of survey, we will review only those aspects of the theory that serve as prerequisites and motivation for the contributed articles. In particular, we will discuss the well-known problem of characterizing compact composition operators on Hardy and related spaces. In Sections 5 to 7 we will describe in more detail the problems studied in the contributed articles and summarize the main results obtained.

2 Nevanlinna counting function and Shapiro's theorem

We recall that for $1 \le p < \infty$ the analytic Hardy space H^p consists of those analytic functions $f\colon \mathbb{D} \to \mathbb{C}$ for which the norm

$$\|f\|_p = \sup_{0<r<1} \left(\int_{\partial\mathbb{D}} |f(r\zeta)|^p \, dm(\zeta) \right)^{1/p} \tag{1}$$

is finite. Here m is the normalized Lebesgue arc-length measure on the unit circle $\partial\mathbb{D}$. According to a variant of Fatou's theorem, each $f \in H^p$ has non-tangential boundary limits $f(\zeta)$ for m-a.e. $\zeta \in \partial\mathbb{D}$, and it is well known that the L^p norm of the boundary function equals (1). In this way H^p can be identified with the closed subspace of $L^p(m)$ consisting of functions whose negative Fourier coefficients are all zero. These and other rudimentary facts about the H^p spaces can be found in any standard reference, e.g. [9, 28].

The class of H^p spaces, and especially the Hilbert space H^2, is the most classical setting for studying composition operators. It is a consequence of *Littlewood's subordination principle* (see e.g. [8, 9]) that every composition operator C_φ restricts to a bounded operator on H^p. In fact, if $\varphi(0) = 0$, then we actually have $\|C_\varphi f\|_p \leq \|f\|_p$ for all $f \in H^p$.

A topic of substantial interest in the literature has been the search for function-theoretic conditions on φ which would characterize when the operator C_φ is compact on H^2.* It is easy to see that this depends on how the values of φ are allowed to approach the boundary $\partial\mathbb{D}$. For instance, if the image $\varphi(\mathbb{D})$ does not touch $\partial\mathbb{D}$ at all, that is, $\|\varphi\|_\infty = \sup\{|\varphi(z)| : z \in \mathbb{D}\} < 1$, then a normal family argument yields that C_φ is compact. Fairly simple examples however show that this sufficient condition is not necessary. At the other extreme, if, say, the set $\Sigma = \{\zeta \in \partial\mathbb{D} : |\varphi(\zeta)| = 1\}$ has a positive Lebesgue measure, then C_φ is non-compact. This can be seen by noting that $z^n \to 0$ weakly in H^2 but $\|\varphi^n\|_2^2 \geq m(\Sigma)$ for all n.

A beautiful solution to the compactness problem was found in 1987 by Joel H. Shapiro [34] in a paper that has stimulated much of the subsequent interest in composition operators. Shapiro made use of the Nevanlinna counting function of φ.

Definition 1. The *Nevanlinna counting function* of a non-constant analytic map $\varphi\colon \mathbb{D} \to \mathbb{D}$ is

$$N_\varphi(w) = \sum_{\varphi(z)=w} -\log|z|, \qquad w \in \mathbb{D} \setminus \{\varphi(0)\},$$

where each pre-image z is counted according to its multiplicity and an empty sum is regarded as zero.

The roots of the counting function lie in Rolf Nevanlinna's renowned theory of value distribution for entire and meromorphic functions (see [24]). The number $N_\varphi(w)$ should be viewed as a measure of the "affinity" that φ has for the value w. It weights each pre-image z by the product of its multiplicity and "logarithmic distance" $-\log|z|$ from the boundary. So pre-images that are located deep inside $\mathbb{D}$ count more than those near $\partial\mathbb{D}$. Since we are mainly interested in values close to the boundary (whose pre-images also lie near $\partial\mathbb{D}$ by the Schwarz lemma), it is reasonable to think of $-\log|z|$ as approximately the Euclidean distance $1 - |z|$.

*Recall that a linear operator from a Banach space X into another Banach space Y is compact (resp. weakly compact) if the image of the unit ball of X is relatively compact (resp. weakly compact) in Y.

A classical result involving N_φ is the inequality

$$N_\varphi(w) \le \log\left|\frac{1-\overline{\varphi(0)}w}{w-\varphi(0)}\right|,$$

which can be traced back to Littlewood. It implies that $N_\varphi(w) = O(-\log|w|)$, and in the special case $\varphi(0) = 0$ it reduces to $N_\varphi(w) \le -\log|w|$, which is actually an improvement of the Schwarz lemma. Shapiro's characterization for the compactness of C_φ is a little-oh version of this inequality. In fact, Shapiro was able to calculate the *essential norm* of C_φ (its distance, in the operator norm, from the compact operators on H^2), usually denoted by $\|C_\varphi\|_e$.

Theorem 2 (Shapiro [34])**.** *Let $\varphi\colon \mathbb{D} \to \mathbb{D}$ be an analytic map. Then C_φ as an operator on H^2 satisfies*

$$\|C_\varphi\|_e^2 = \limsup_{|w|\to 1} \frac{N_\varphi(w)}{-\log|w|}.$$

Hence C_φ is compact on H^2 if and only if $N_\varphi(w) = o(-\log|w|)$ as $|w| \to 1$.

A straightforward argument based on the canonical inner-outer factorization of H^p functions shows that C_φ is compact on any H^p with $1 \le p < \infty$ if and only if it is compact on H^2 (see [39]). Thus Shapiro's compactness criterion works for all these spaces. Furthermore, it was later shown by Donald Sarason [31] that on the non-reflexive space H^1 every weakly compact composition operator is compact and hence admits the same characterization.

For later reference we next record a few ingredients of Shapiro's proof for Theorem 2. His basic tool is the identity

$$\|C_\varphi f\|_2^2 = |f(\varphi(0))|^2 + 2\int_{\mathbb{D}} |f'|^2 N_\varphi\, dA, \tag{2}$$

where $f \in H^2$ and A is the Lebesgue area measure normalized so that $A(\mathbb{D}) = 1$. This identity establishes a direct connection between the composition operator and the Nevanlinna counting function. Shapiro calls it a *change-of-variable formula* as it can be derived by a change of variables from the classical Littlewood–Paley identity

$$\|f\|_2^2 = |f(0)|^2 + 2\int_{\mathbb{D}} |f'(z)|^2(-\log|z|)\, dA(z).$$

As Shapiro points out, (2) is also a special case of C. S. Stanton's general formula for integral means of subharmonic functions [10, 40, 41].

In the course of establishing the lower bound for $\|C_\varphi\|_e$ Shapiro employs the test functions $f_a(z) = \sqrt{1-|a|^2}/(1-\bar{a}z)$, which are the normalized reproducing kernels for H^2. They have the property that $\|f_a\|_2 = 1$ for all $a \in \mathbb{D}$ but $f_a \to 0$ weakly as $|a| \to 1$. In fact, Shapiro's proof shows that

$$\|C_\varphi\|_e = \limsup_{|a|\to 1} \|C_\varphi f_a\|_2. \tag{3}$$

We will refer to this fact at the end of Section 4.

Remark 3. For the history of the compactness problem and developments that led to Theorem 2 we refer to the monographs [8, 36] and the references mentioned therein. Another widely-used approach to the boundedness and compactness of composition operators on Hardy spaces and their relatives is based on Carleson-type measure considerations; see e.g. Section 3.2 of [8].

3 Aleksandrov measures

Aleksandrov measures are a fascinating concept. They emerge in a natural way and play a significant role in many areas of mathematical analysis. In this section we will review some of their basic properties from a function-theoretic point of view. In the next section we will explain how they interact with composition operators.

Let us start with the definition. Here and throughout the text we use P_z to denote the Poisson kernel for $z \in \mathbb{D}$; that is,

$$P_z(\zeta) = \frac{1-|z|^2}{|\zeta - z|^2}, \qquad \zeta \in \partial\mathbb{D}.$$

Definition 4. Let $\varphi\colon \mathbb{D} \to \mathbb{D}$ be an analytic map. For every $\alpha \in \partial\mathbb{D}$ let τ_α be the positive Borel measure on $\partial\mathbb{D}$ such that

$$\operatorname{Re} \frac{\alpha + \varphi(z)}{\alpha - \varphi(z)} = \frac{1 - |\varphi(z)|^2}{|\alpha - \varphi(z)|^2} = \int_{\partial\mathbb{D}} P_z \, d\tau_\alpha$$

for all $z \in \mathbb{D}$. Then the measures τ_α are called the *Aleksandrov measures* of φ.

The definition makes sense because the left-hand side is a positive harmonic function on $\mathbb{D}$ and every such function can be represented as the Poisson integral of a unique positive measure. Instead of the Poisson representation one can invoke the Herglotz formula to write

$$\frac{\alpha + \varphi(z)}{\alpha - \varphi(z)} = \int_{\partial\mathbb{D}} \frac{\zeta + z}{\zeta - z} \, d\tau_\alpha(\zeta) + i t_\alpha,$$

where t_α is the imaginary part of $(\alpha+\varphi(0))/(\alpha-\varphi(0))$. In the reverse direction, if α is given and μ is any positive and finite Borel measure on $\partial\mathbb{D}$, one can use this formula to construct a map φ whose Aleksandrov measure τ_α equals μ.

The name of the measures is after Alexei B. Aleksandrov, who has derived many of their deep properties and used them as a valuable tool in the context of classical harmonic analysis, for example in studying the boundary values of inner functions [1]. We will discuss some of his ideas in a moment. Elsewhere in the literature Aleksandrov measures are often referred to as "Clark measures" or "spectral measures". This is due to Douglas N. Clark's seminal paper [6], which was the first to really call attention to these measures. There Clark showed that the Aleksandrov measures corresponding to a certain inner function arise as spectral measures of unitary rank-one perturbations of the so-called model operator.

The theory that has developed around Clark's and Aleksandrov's fundamental discoveries is nowadays extensive, ranging from analytic function theory to questions in mathematical physics. For more information and pertinent references we refer the reader to the recent lecture notes [29] and to the surveys [19, 26]. Also the book [5] contains an accessible account on the topic.

Some elementary properties of Aleksandrov measures are collected in the next proposition. They all follow easily from the rudiments of Poisson integral representations (see e.g. [9, 28]) when applied to Definition 4.

Proposition 5. *Let $\varphi\colon \mathbb{D} \to \mathbb{D}$ be an analytic map and let τ_α ($\alpha \in \partial\mathbb{D}$) be the associated Aleksandrov measures with Lebesgue decompositions $\tau_\alpha = \tau_\alpha^a + \tau_\alpha^s$ with respect to m. Then:*

a) The total mass of τ_α is

$$\|\tau_\alpha\| = \frac{1 - |\varphi(0)|^2}{|\alpha - \varphi(0)|^2}.$$

In particular, τ_α is a probability measure if $\varphi(0) = 0$.

b) The absolutely continuous component of τ_α is

$$\tau_\alpha^a(\zeta) = \frac{1 - |\varphi(\zeta)|^2}{|\alpha - \varphi(\zeta)|^2},$$

where $\varphi(\zeta)$ are the non-tangential boundary limits of φ (defined and different from α for m-a.e. $\zeta \in \partial\mathbb{D}$). Hence the measures τ_α^a are mutually absolutely continuous.

c) The singular component τ_α^s is carried by the set where $\varphi(\zeta) = \alpha$. Hence the measures τ_α^s are mutually singular.

d) τ_α is singular if and only φ is an inner function, i.e. $|\varphi(\zeta)| = 1$ for m-a.e. $\zeta \in \partial\mathbb{D}$.

For the purposes of the present dissertation, especially Article [A], the most important aspects of the Aleksandrov measures are related to the boundary behaviour of the inducing map. Here the singular components τ_α^s play a crucial role. Indeed, in view of part (c) of the above proposition, the presence of singularity in τ_α indicates that the map φ assumes α as its boundary value. Moreover, the location and magnitude of the singular mass reflect the local "affinity" of φ for α in a very natural way. To illustrate this we consider the particular case of point masses, or atoms.

Let us recall that an analytic map $\varphi\colon \mathbb{D} \to \mathbb{D}$ is said to have a *finite angular derivative* at a point $\zeta \in \partial\mathbb{D}$ if φ has a unimodular boundary value at ζ and the difference quotient $(\varphi(z) - \varphi(\zeta))/(z - \zeta)$ tends to a finite limit as $z \to \zeta$ non-tangentially. The limit, which can be denoted $\varphi'(\zeta)$, is called the *angular derivative* of φ at ζ. It turns out that the angular derivatives of φ have a perfect correspondence with the atoms of the Aleksandrov measures.

Proposition 6. *Let $\varphi\colon \mathbb{D} \to \mathbb{D}$ be an analytic map and $\zeta \in \partial\mathbb{D}$. Then φ has a finite angular derivative at ζ if and only if there is $\alpha \in \partial\mathbb{D}$ such that the Aleksandrov measure τ_α has an atom at ζ. Furthermore, in that case $\varphi(\zeta) = \alpha$ and $|\varphi'(\zeta)| = \tau_\alpha(\{\zeta\})^{-1}$.*

For the proof of this proposition convenient references are [11, 29], where it is established in conjunction with the classical Julia–Carathéodory theorem. The idea of relating angular derivatives to point masses actually goes back to Nevanlinna's note [23] from 1929, which can perhaps be seen as the first genuine function-theoretic application of the Aleksandrov measures.

Our philosophy of viewing the measures τ_α^s as measuring the boundary affinity of φ is in line with Aleksandrov's ideas in [1]. Aleksandrov was concerned with the notion of multiplicity for unimodular boundary values of analytic self-maps of the disc, especially inner functions. According to his definition, the *multiplicity* of a boundary value $\alpha \in \partial\mathbb{D}$ for φ is the cardinality of the (measure-theoretic) support of τ_α^s. Therefore it makes sense to speak about boundary values of finite, countable or continuum multiplicity. Clearly, the multiplicity of α is finite or countable if and only if the measure τ_α^s is discrete, i.e. consists of atoms only. Aleksandrov's work exhibits interesting examples of functions with various types of boundary values.

In [1] Aleksandrov also established a curious disintegration formula involving the family of measures τ_α. For every continuous function $f\colon \partial\mathbb{D} \to \mathbb{C}$ it asserts that

$$\int_{\partial\mathbb{D}} f\,dm = \int_{\partial\mathbb{D}} \left(\int_{\partial\mathbb{D}} f\,d\tau_\alpha \right) dm(\alpha). \tag{4}$$

Indeed, if f is a Poisson kernel, this is an immediate consequence of the definition of the measures τ_α and the fact that the integral of a Poisson kernel is 1. The general case follows by approximation. Hence the family of τ_α:s "disintegrates" the Lebesgue measure in a way that can be written as

$$m = \int_{\partial\mathbb{D}} \tau_\alpha\,dm(\alpha)$$

in the weak* sense of measures. Furthermore, it is not difficult to see that the singular components τ_α^s correspond to unimodular boundary values of φ in the sense that $\chi_\Sigma m = \int \tau_\alpha^s\,dm(\alpha)$, where $\Sigma = \{\zeta \in \partial\mathbb{D} : |\varphi(\zeta)| = 1\}$.

Formula (4) also leads one to consider the linear operator A_φ given by

$$A_\varphi f(\alpha) = \int_{\partial\mathbb{D}} f\,d\tau_\alpha.$$

It takes continuous functions to continuous functions and is often called the *Aleksandrov operator*. We will see in the next section that this operator bears a close relationship to composition operators. In [1] Aleksandrov extended the definition of A_φ as well as the scope of (4) to all Lebesgue integrable functions f on $\partial\mathbb{D}$.

4 Sarason's approach to composition operators

In 1990 Donald Sarason [30] introduced a novel approach to the study of composition operators. His idea was to view C_φ as an integral operator acting on the unit circle $\partial\mathbb{D}$. This is accomplished basically via Poisson extension. Although Sarason did not make an explicit reference to Aleksandrov measures, it is not difficult to reformulate his ideas as using them. This will establish an elegant connection between composition operators and Aleksandrov measures, which is essential for Article [B].

Let M denote the space of all complex Borel measures on $\partial\mathbb{D}$. Whenever $\mu \in M$, the Poisson integral $u(z) = \int P_z \, d\mu$ defines a harmonic function u on $\mathbb{D}$. In fact, it is well known that this determines an isometric isomorphism from M onto the *harmonic Hardy space* h^1 of all harmonic functions $u\colon \mathbb{D} \to \mathbb{C}$ for which the norm $\|u\|_1$ defined as in (1) is finite. Thus, if u corresponds to μ as above and $\varphi\colon \mathbb{D} \to \mathbb{D}$ is an analytic map, the composition $v = u \circ \varphi$ is also harmonic in $\mathbb{D}$. Since μ can be expressed as a linear combination of positive measures and every positive harmonic function belongs to h^1, it follows that v is the Poisson integral of a unique measure $\nu \in M$. Hence it is natural to define $C_\varphi \mu = \nu$. Sarason observed that the resulting linear operator C_φ acts boundedly on M, and if L^1 is the space of Lebesgue integrable functions (or measures absolutely continuous with respect to m), then C_φ maps L^1 into itself.

To understand the action of C_φ on the unit circle, it is worthwhile to note that the correspondence $C_\varphi \mu = \nu$ can be written as

$$\int_{\partial\mathbb{D}} P_z \, d\nu = \int_{\partial\mathbb{D}} P_{\varphi(z)} \, d\mu \tag{5}$$

for all $z \in \mathbb{D}$. In particular, if μ is the unit mass δ_α for some $\alpha \in \partial\mathbb{D}$, we get the definition of the Aleksandrov measure τ_α associated with φ. Thus $C_\varphi \delta_\alpha = \tau_\alpha$. Approximating a continuous function $f\colon \partial\mathbb{D} \to \mathbb{C}$ by linear combinations of Poisson kernels we also arrive at the identity

$$\int_{\partial\mathbb{D}} f \, d\nu = \int_{\partial\mathbb{D}} \left(\int_{\partial\mathbb{D}} f \, d\tau_\alpha \right) d\mu(\alpha),$$

which is an extension of Aleksandrov's disintegration formula (4). Hence the definition of C_φ on M can be rephrased in the weak* sense of measures as

$$C_\varphi \mu = \int_{\partial\mathbb{D}} \tau_\alpha \, d\mu(\alpha).$$

In other words, C_φ is the adjoint of the Aleksandrov operator A_φ introduced at the end of Section 3.

The above reasoning shows that the composition operator C_φ can be regarded as an integral operator whose (measure-valued) kernel is provided by the family of Aleksandrov measures τ_α. Thus it becomes feasible to relate the operator-theoretic properties of C_φ to the properties of the kernel. This is what Sarason did. His main results address the compactness and weak compactness of C_φ as an operator on M and L^1, and they can be summarized as follows.

Theorem 7 (Sarason [30])**.** *Let $\varphi\colon \mathbb{D} \to \mathbb{D}$ be an analytic map and let τ_α ($\alpha \in \partial\mathbb{D}$) be the associated Aleksandrov measures. Then the following conditions are equivalent:*

a) C_φ is (weakly) compact on M.
b) C_φ is (weakly) compact on L^1.
c) For each α, the measure τ_α is absolutely continuous, i.e. $\tau_\alpha^s = 0$.

Sarason later called (c) the *absolute continuity condition* (see Section XI-4 of [32]). In view of Proposition 5 it can be written as

$$\int_{\partial\mathbb{D}} \frac{1-|\varphi(\zeta)|^2}{|\alpha-\varphi(\zeta)|^2}\, dm(\zeta) = \frac{1-|\varphi(0)|^2}{|\alpha-\varphi(0)|^2}$$

for all $\alpha \in \partial\mathbb{D}$.

Given Theorems 2 and 7 it is natural to wonder the relationship between Shapiro's little-oh condition for the Nevanlinna counting function and Sarason's absolute continuity condition. Since H^1 can be viewed as a subspace of L^1 (corresponding to functions whose negative Fourier coefficients vanish), it is clear that Sarason's condition implies Shapiro's. The converse was established by Shapiro together with Carl Sundberg [37]. They showed that the little-oh condition for the counting function of φ implies that C_φ is compact on M.

In summary: C_φ is compact on any of the spaces H^p $(1 \le p < \infty)$, L^1 and M or weakly compact on H^1, L^1 and M if and only if the associated Aleksandrov measures are all absolutely continuous, or, equivalently, the Nevanlinna counting function satisfies $N_\varphi(w) = o(-\log|w|)$ as $|w| \to 1$.

The connection between Shapiro's and Sarason's compactness conditions was enhanced by Joseph A. Cima and Alec L. Matheson [4]. They proved the following.

Theorem 8 (Cima–Matheson [4])**.** *Let $\varphi\colon \mathbb{D} \to \mathbb{D}$ be an analytic map and let τ_α ($\alpha \in \partial\mathbb{D}$) be the associated Aleksandrov measures. Then C_φ as an operator on H^2 satisfies*

$$\|C_\varphi\|_e^2 = \sup_{\alpha\in\partial\mathbb{D}} \|\tau_\alpha^s\|.$$

The argument of Cima and Matheson is very elegant and we reproduce it here. Let us consider the normalized kernel functions $f_a(z) = \sqrt{1-|a|^2}/(1-\overline{a}z)$ (cf. the end of Section 2) and write

$$|C_\varphi f_a|^2 = \frac{1-|a|^2}{|1-\overline{a}\varphi|^2} = \frac{1-|\overline{a}\varphi|^2}{|1-\overline{a}\varphi|^2} - |a|^2\frac{1-|\varphi|^2}{|1-\overline{a}\varphi|^2}.$$

To obtain $\|C_\varphi f_a\|_2^2$ we integrate this decomposition with respect to m. The first term on the right-hand side is bounded and harmonic in $\mathbb{D}$, so its integral equals $(1-|\overline{a}\varphi(0)|^2)/|1-\overline{a}\varphi(0)|^2$, which converges to $\|\tau_\alpha\|$ as $a \to \alpha$. Since the second term tends pointwise to $(1-|\varphi|^2)/|\alpha-\varphi|^2 = \tau_\alpha^a$, an application of Fatou's lemma yields that $\limsup_{a\to\alpha}\|C_\varphi f_a\|_2^2 \le \|\tau_\alpha\| - \|\tau_\alpha^a\| = \|\tau_\alpha^s\|$. Moreover, if $a \to \alpha$ radially, then the second term is actually increasing and we have $\|C_\varphi f_a\|_2^2 \to \|\tau_\alpha^s\|$. In view of equation (3), this proves Theorem 8.

Remark 9. The relation between the normalized reproducing kernels and the Aleksandrov measures was further refined by Jonathan E. Shapiro [33]. He observed that

$$|C_\varphi f_a|^2 \to \tau_\alpha^s \quad \text{weak*} \qquad \text{as } a \to \alpha \text{ non-tangentially,}$$

when $|C_\varphi f_a|^2$ is regarded as a measure on $\partial\mathbb{D}$.

Remark 10. In a forthcoming paper [25] the author has extended Theorem 7 by showing that the essential norm of C_φ as an operator on M and L^1 equals $\sup\{\|\tau_\alpha^s\| : \alpha \in \partial\mathbb{D}\}$. The same result is true for the weak essential norm (i.e. distance from weakly compact operators) as well.

5 Summary of [A]: Boundary correspondence of Nevanlinna counting functions

In Section 2 we defined the classical Nevanlinna counting function $N_\varphi(w) = \sum\{-\log|z| : \varphi(z) = w\}$ of a non-constant analytic map $\varphi\colon \mathbb{D} \to \mathbb{D}$. Then $N_\varphi(w)$ is a quantity measuring the (total) affinity of φ for a value $w \in \mathbb{D}$. On the other hand, in Section 3 we have viewed the singular components of the Aleksandrov measures as measuring the affinity of φ for a boundary value $\alpha \in \partial\mathbb{D}$. This line of thought naturally raises the following question: Is there any exact relation between the family of singular measures τ_α^s and the behaviour of $N_\varphi(w)$ as w tends to the boundary?

Strong evidence in the positive direction comes from the theory of composition operators. Theorems 2 and 8 gave two rather different expressions for the essential norm of the operator C_φ on H^2. The equality of these expressions shows that there is a connection between the decay rate of N_φ at the boundary and the maximal size of the measures τ_α^s:

$$\limsup_{|w|\to 1} \frac{N_\varphi(w)}{-\log|w|} = \sup_{\alpha\in\partial\mathbb{D}} \|\tau_\alpha^s\|. \tag{6}$$

However, the theory of composition operators as such does not give any function-theoretic explanation for this equality.

Our goal in Article [A] is to establish a definite local boundary correspondence between the Nevanlinna counting function and the singular components of the Aleksandrov measures. In particular, our result will explain the precise analytic mechanism underlying the somewhat mysterious equality (6).

As a first step towards establishing such a correspondence one might conjecture that

$$\frac{N_\varphi(w)}{-\log|w|} \to \|\tau_\alpha^s\| \tag{7}$$

as $w \to \alpha$. However, fairly simple examples show that one has to impose some restrictions on the region of approach to the boundary; even a non-tangential version of (7) may fail unless we allow some exceptional sets.

Example 11. Let $\varphi(z) = \frac{1}{2}(1+z)$. Since $\varphi(1) = 1$ and $\varphi'(1) = \frac{1}{2}$, it follows by Proposition 6 that $\tau_1^s = 2\delta_1$. But $N_\varphi(w) = 0$ for w outside the range of φ, which is the open disc with centre and radius equal to $\frac{1}{2}$. Thus (7) does not hold as $w \to 1$ unrestrictedly in $\mathbb{D}$.

Example 12. Let (a_n) be any sequence in $\mathbb{D}$ with $a_n \to 1$, and let Ω be the domain formed from $\mathbb{D}$ by deleting the points a_n. Let φ be an analytic covering map from $\mathbb{D}$ onto Ω. A well-known property of covering maps is that the boundary limits $\varphi(\zeta)$ (which, by Fatou's theorem, exist for m-a.e. ζ) all lie in $\partial\Omega$. Since the set where $\varphi(\zeta) = a_n$ has measure zero for every n, it follows that φ is an inner function. Hence its Aleksandrov measures are all singular, and so $\|\tau_1^s\| = (1 - |\varphi(0)|^2)/|1 - \varphi(0)|^2$. But clearly $N_\varphi(a_n) = 0$ for all n.

Another issue to be resolved is that we would like to be able to catch the whole measure τ_α^s, not only its total mass. To this end we introduce a measure-valued version of the counting function so as to distinguish between different pre-images of a given point w.

Definition 13. The *Nevanlinna counting measure* of φ is

$$M_\varphi(w) = \frac{1}{-\log|w|} \sum_{\varphi(z)=w} (-\log|z|)\, \delta_z, \qquad w \in \mathbb{D} \setminus \{\varphi(0)\},$$

where δ_z is the unit mass at z. As in the definition of N_φ, the sum takes account of multiplicities and an empty sum is regarded as zero. Also note that the total mass of $M_\varphi(w)$ equals $N_\varphi(w)/(-\log|w|)$.

Our main result in [A] is the following.

Theorem 14 ([A])**.** *Let $\varphi\colon \mathbb{D} \to \mathbb{D}$ be a non-constant analytic map and fix $\alpha \in \partial\mathbb{D}$. Let τ_α be the Aleksandrov measure of φ at α. Then*

$$M_\varphi(w) \to \tau_\alpha^s \quad \textit{weak*}$$

as $w \to \alpha$ non-tangentially off an exceptional set of finite Green capacity.

The weak* convergence should be understood in the space of Borel measures on $\overline{\mathbb{D}}$ (as the dual of continuous functions on $\overline{\mathbb{D}}$). Note in particular that it implies the convergence of total masses because the measures involved are positive.

The notion of Green capacity is quite similar to that of logarithmic capacity, which is commonly used in complex analysis; instead of the logarithmic kernel $-\log|z-w|$ we just use the Green function for the disc (see e.g. [11, 27]). This makes the Green capacity conformally invariant. To elucidate the capacity condition in Theorem 14 we remark that if $E \subset \mathbb{D}$ is a set of finite Green capacity and $\alpha \in \partial\mathbb{D}$, then the capacity of $\{w \in E : |w - \alpha| < r\}$ tends to zero as $r \to 0$. So the set E looks very "thin" near the boundary point α. In particular, it cannot contain any continuum that would join a point of $\mathbb{D}$ to α. Hence it is always possible to find sequences (r_n) increasing to 1 such that $M_\varphi(r_n\alpha) \to \tau_\alpha^s$ weak*.

Remark 15. The capacity estimate given in Theorem 14 is, in general, the best possible. This is shown in Section 8 of [A]. The idea is to construct an analytic covering map as in Example 12 above; instead of the points a_n we now delete a sequence of small closed discs converging to 1. It is possible to arrange these in such a way that $\tau_1^s \neq 0$ but the Green capacity of the set $\{w \in \mathbb{D} \setminus \varphi(\mathbb{D}) : |w - 1| < r\}$ decays to zero as slowly as we wish when $r \to 0$.

Nevertheless, in some special cases Theorem 14 can be strengthened considerably. For instance, assume that τ_α^s is a discrete measure (i.e. consists of point masses only), which is the case if e.g. φ is boundedly valent. Then the weak* convergence of $M_\varphi(w)$ to τ_α^s takes place non-tangentially, without any exceptional set. The proof of this result (see Theorem 10.3 of [A]) is essentially due to Shapiro [34], and it is based on the fact that the atoms of τ_α correspond to finite angular derivatives of φ (see Proposition 6).

We want to emphasize that Theorem 14 should be seen as a purely function-theoretic statement that forges a link between the value distribution of φ inside the disc and on its boundary. Yet its proof makes substantial use of tools and ideas coming from the theory of composition operators. We close this section by giving a very rough outline of our argument.

We start by considering the convergence of total masses as in (7). Here we utilize the normalized kernel functions f_a and combine the Cima–Matheson identity $\|\tau_\alpha^s\| = \lim_{r\to 1}\|C_\varphi f_{r\alpha}\|_2^2$ (see the end of Section 4) with the change-of-variable formula (2) to obtain

$$\|\tau_\alpha^s\| = \lim_{r\to 1} 2\int_{\mathbb{D}} |f'_{r\alpha}(w)|^2 N_\varphi(w)\, dA(w).$$

Given a non-tangential approach region Γ for the point α, it is now relatively easy to find a sequence (w_n) converging in Γ to α such that (7) holds true along that sequence. Moreover, the sequence can be chosen in such a way that the hyperbolic distances between its successive members stay bounded. Thus we can cover Γ by hyperbolic discs centred at points w_n and having bounded radii. In each disc we then do careful potential-theoretic analysis based on the subharmonicity property of N_φ, and eventually arrive at a global estimate showing that $N_\varphi(w)/(-\log|w|)$ stays close to $\|\tau_\alpha^s\|$ outside a set of finite Green capacity. The final step of the proof involves a change of variables trick which shows, perhaps surprisingly, that the convergence of total masses of counting measures actually guarantees the full weak* convergence as required by Theorem 14.

6 Summary of [B]: Compact differences of composition operators on Hardy and Lebesgue spaces

When X is a Banach space of analytic or harmonic functions on $\mathbb{D}$, we let $\mathcal{C}(X)$ denote the set of bounded composition operators on X. An important theme in the study of composition operators has been the inquiry into the topological structure of the set $\mathcal{C}(X)$ for various choices of X. In addition to the usual topology induced by the operator norm, one may equip $\mathcal{C}(X)$ with other topologies, such as the one given by the essential norm (which is equivalent to considering the structure of $\mathcal{C}(X)$ relative to the Calkin algebra on X). In this connection it becomes essential to study the mapping properties of differences of composition operators, i.e. operators of the form $C_\varphi - C_\psi$.

In the case of H^2 this line of research was initiated in Earl Berkson's note [2] and continued in the remarkable papers of Barbara MacCluer [15] and Shapiro and Sundberg [38]. Shapiro and Sundberg, in particular, were concerned with the task of determining the isolated members of $\mathcal{C}(H^2)$. They presented a number of theorems and examples that relate the isolation of C_φ to the "order of contact" the symbol φ has with the unit circle. Towards the end of their paper they also raised the following two questions:

- What are the components of $\mathcal{C}(H^2)$ like?
- Which composition operators have a compact difference on H^2?

Both questions have proved quite intractable and still lack complete solutions. Nevertheless, they have inspired a great deal of interesting research, also in settings other than H^2.

In Article [B] we consider the problem of compact differences on the H^p spaces and the closely related spaces L^1 and M. As a main tool we use Aleksandrov measures. In fact, one of the purposes of [B] is to investigate to what extent the absolute continuity criterion for the compactness of a single composition operator can be adapted to the case of differences. Let us recall here that by the results of Sarason, Shapiro and Sundberg (cf. Section 4) the absolute continuity of the Aleksandrov measures characterizes the compactness of C_φ on H^p ($1 \le p < \infty$), L^1 and M, and the weak compactness on H^1, L^1 and M.

The idea of using Aleksandrov measures to examine differences of composition operators is not new. It was introduced in the case of H^2 by Jonathan E. Shapiro [33], who extended some results of MacCluer [15]. MacCluer had shown, in particular, that a necessary condition for the difference $C_\varphi - C_\psi$ to be compact is that the angular derivatives of φ and ψ coincide; that is, the Aleksandrov measures of φ and ψ must have identical atoms. J. E. Shapiro improved this as follows.

Theorem 16 (J. E. Shapiro [33])**.** *Let $\varphi, \psi \colon \mathbb{D} \to \mathbb{D}$ be analytic maps and let $\mu_\alpha = \mu_\alpha^a + \mu_\alpha^s$ and $\nu_\alpha = \nu_\alpha^a + \nu_\alpha^s$ be the Lebesgue decompositions of the associated Aleksandrov measures. If $C_\varphi - C_\psi$ is compact on H^2, then $\mu_\alpha^s = \nu_\alpha^s$ for all $\alpha \in \partial\mathbb{D}$.*

J. E. Shapiro also conjectured that the converse would be true. In Section 5 of [B] we observe that this is not the case.

Theorem 17 ([B]). *There exist univalent maps $\varphi, \psi \colon \mathbb{D} \to \mathbb{D}$ such that $C_\varphi - C_\psi$ is non-compact on H^2, but $\mu_1^s = \nu_1^s = c\delta_1$ for some $c > 0$ and $\mu_\alpha^s = \nu_\alpha^s = 0$ for $\alpha \neq 1$.*

Theorem 17 indicates that in addition to the equality of the singular components of the Aleksandrov measures some other condition is needed to make the difference $C_\varphi - C_\psi$ compact on H^2. One such condition is provided in Theorem 19 below, but, as it turns out, it is not necessary for the case of H^2. Apparently, the characterization of compactness of $C_\varphi - C_\psi$ on H^2 is one of the most famous open problems in the field of composition operators. Very interesting progress in this area has recently been made by T. Kriete, J. Moorhouse and C. Toews [14, 21, 22]. In particular, Moorhouse [21] found a characterization for the compactness of $C_\varphi - C_\psi$ in the related setting of (weighted) Bergman spaces.

For the rest of the H^p spaces with $1 \leq p < \infty$ we have the following result.

Theorem 18 ([B]). *Let $\varphi, \psi \colon \mathbb{D} \to \mathbb{D}$ be analytic maps and $T = C_\varphi - C_\psi$. If $1 \leq p < \infty$, then the following conditions are equivalent:*

a) T is compact on H^2.
b) T is compact on H^p.
c) T is weakly compact on H^1.

Most of Theorem 18 follows by interpolation. We also apply a factorization trick to show that compactness on H^2 implies compactness on H^1. Moreover, to deal with (c) we use a modification of Sarason's argument for a single composition operator [31].

We proceed to consider the cases of L^1 and M. Here we can give a complete characterization for compact and weakly compact differences. To this end recall that a set $F \subset L^1$ is *uniformly integrable* if for every $\epsilon > 0$ there exists a number L such that

$$\int_{\{|f| > L\}} |f| \, dm \leq \epsilon \quad \text{for all } f \in F.$$

The notation in the next theorem is the same as in Theorem 16 above.

Theorem 19 ([B]). *Let $\varphi, \psi \colon \mathbb{D} \to \mathbb{D}$ be analytic maps and $T = C_\varphi - C_\psi$. Then the following conditions are equivalent:*

a) T is (weakly) compact on M.
b) T is (weakly) compact on L^1.
c) $\mu_\alpha^s = \nu_\alpha^s$ for all $\alpha \in \partial\mathbb{D}$ and the set $\{\mu_\alpha^a - \nu_\alpha^a : \alpha \in \partial\mathbb{D}\}$ is uniformly integrable.

A key tool in the proof of this result is the classical Dunford–Pettis theorem (see e.g. [42]) which says that a set in L^1 is relatively weakly compact if and only if it is uniformly integrable.

Since H^1 can be regarded as a subspace of L^1, Theorems 17 and 18 show that the uniform integrability condition in part (c) of Theorem 19 cannot be dispensed with. On the other hand, we see that (c) is sufficient for the compactness of T on H^1 and hence on the entire H^p scale for $1 \leq p < \infty$. The converse however fails. This is a consequence of the following somewhat surprising theorem, which is a main result of [B]. Thus the case of differences is not completely parallel with the case of a single composition operator.

Theorem 20 ([B]). *There exist analytic maps $\varphi, \psi \colon \mathbb{D} \to \mathbb{D}$ such that $C_\varphi - C_\psi$ is compact on H^1 but non-compact on L^1.*

The construction is fairly complicated and technical. The map φ is obtained as a covering map onto a rather peculiar domain Ω contained in $\mathbb{D}$ and ψ is defined as a suitable perturbation of φ. The analysis of the compactness properties of $C_\varphi - C_\psi$ then involves, among other things, delicate estimation of the harmonic measure of Ω. It would certainly be desirable to find a simpler proof for Theorem 20.

7 Summary of [C]: Compact differences of composition operators on Bloch and Lipschitz spaces

As we mentioned at the beginning of Section 6, the research initiated by Shapiro and Sundberg on the structure of the set of composition operators on H^2 has served as an impetus for similar studies in other function spaces. An important example is the paper by Barbara MacCluer, Shûichi Ohno and Ruhan Zhao [16], in which the problems of component structure and compact differences were solved for H^∞, the space of bounded analytic functions under the supremum norm. For instance, it was proved in [16] that a difference operator $C_\varphi - C_\psi$ is compact on H^∞ if and only if

$$\rho\big(\varphi(z), \psi(z)\big) \to 0 \quad \text{as } \max\big(|\varphi(z)|, |\psi(z)|\big) \to 1, \tag{8}$$

where $\rho(z, w) = |z - w|/|1 - \overline{w}z|$ is the pseudo-hyperbolic distance in $\mathbb{D}$. It is not difficult to show that (8) characterizes weak compactness as well (see e.g. [B]).

The ideas of [16] have subsequently been extended to various directions. In particular, Takuya Hosokawa and Shûichi Ohno [12, 13] have recently addressed the same problems for the classical Bloch and little Bloch spaces. In Article [C] our purpose is to complement and extend their results on compact differences. For $0 < \alpha \leq 1$ we consider the Bloch-type space $\mathcal{B}^\alpha$ of analytic functions $f \colon \mathbb{D} \to \mathbb{C}$ whose derivative satisfies the growth condition

$$\sup_{z \in \mathbb{D}} (1 - |z|^2)^\alpha |f'(z)| < \infty.$$

This is a Banach space under the norm obtained by adding, say, $|f(0)|$ to the supremum above. Of course, $\mathcal{B}^1$ is just the standard *Bloch space*, denoted by $\mathcal{B}$. For $0 < \alpha < 1$ it follows from a result of Hardy and Littlewood (see e.g. Theorem 5.1 of [9]) that $\mathcal{B}^\alpha$ can be identified with the analytic *Lipschitz space of order* $1 - \alpha$. From a functional-analytic viewpoint these spaces have been treated in [43].

Before explaining our results, we fix some notation and recall a few facts about composition operators on the spaces $\mathcal{B}^\alpha$. When φ is an analytic self-map of $\mathbb{D}$, it will be convenient to use the notation

$$\mathcal{D}^\alpha\varphi(z) = \left(\frac{1-|z|^2}{1-|\varphi(z)|^2}\right)^\alpha \varphi'(z),$$

and in the Bloch case we simply write $\mathcal{D}\varphi$ for $\mathcal{D}^1\varphi$. Then $\mathcal{D}^\alpha\varphi$ can be regarded as a derivative of φ relative to the metric induced in $\mathbb{D}$ by the arc-length element $(1-|w|^2)^{-\alpha}|dw|$. In particular, $\mathcal{D}\varphi$ is the hyperbolic derivative of φ.

In the Lipschitz case $0 < \alpha < 1$ it was shown by K. M. Madigan [17] that C_φ acts boundedly on $\mathcal{B}^\alpha$ if and only if $\mathcal{D}^\alpha\varphi$ is bounded, i.e. $\|\mathcal{D}^\alpha\varphi\|_\infty < \infty$. In the Bloch case $\alpha = 1$ this condition is always true by the Schwarz–Pick lemma, so every composition operator is bounded on $\mathcal{B}$. This was observed by Madigan and Matheson [18], who also showed that C_φ is (weakly) compact on $\mathcal{B}$ if and only if $\mathcal{D}\varphi(z) \to 0$ as $|\varphi(z)| \to 1$. An analogous compactness characterization in terms of $\mathcal{D}^\alpha\varphi$ applies in the Lipschitz case; however, it follows from a general result due to J. H. Shapiro [35] that it actually reduces to the conditions $\varphi \in \mathcal{B}^\alpha$ and $\|\varphi\|_\infty < 1$. (For a function-theoretic explanation of this equivalence, see Section 5 of [C].)

Our first theorem in [C] characterizes compact and weakly compact differences of composition operators on the standard Bloch space.

Theorem 21 ([C])**.** *Let $\varphi, \psi\colon \mathbb{D} \to \mathbb{D}$ be analytic maps. Then $C_\varphi - C_\psi$ is* (*weakly*) *compact on $\mathcal{B}$ if and only if the following two conditions hold:*

$$\begin{aligned} \mathcal{D}\varphi(z)\,\rho\big(\varphi(z),\psi(z)\big) &\to 0 \quad \textit{as } |\varphi(z)| \to 1,\\ \mathcal{D}\psi(z)\,\rho\big(\varphi(z),\psi(z)\big) &\to 0 \quad \textit{as } |\psi(z)| \to 1. \end{aligned}$$

This is an improvement to a result of Hosokawa and Ohno [12, 13] in which it was also required that

$$\mathcal{D}\varphi(z) - \mathcal{D}\psi(z) \to 0 \quad \text{as } \min\big(|\varphi(z)|, |\psi(z)|\big) \to 1. \tag{9}$$

Our contribution is to show that this third condition is actually implied by the other two, so it can be dispensed with.

The conditions of Theorem 21 intertwine in a natural way the Madigan–Matheson compactness criterion for a single composition operator and condition (8) for the H^∞ case. In particular, we see that (8) is sufficient for compactness in the Bloch case. This already follows from [16] since it was shown there to characterize when $C_\varphi - C_\psi$ is compact as an operator from $\mathcal{B}$ to H^∞.

Our second theorem deals with the Lipschitz case $0 < \alpha < 1$. Here we assume that the $\mathcal{D}^\alpha$ derivatives of the symbols are bounded in order to ensure the boundedness of the induced composition operators.

Theorem 22 ([C])**.** *Let $0 < \alpha < 1$ and let $\varphi, \psi \colon \mathbb{D} \to \mathbb{D}$ be analytic maps with $\|\mathcal{D}^\alpha \varphi\|_\infty < \infty$ and $\|\mathcal{D}^\alpha \psi\|_\infty < \infty$ so that C_φ and C_ψ are bounded on $\mathcal{B}^\alpha$. Then $C_\varphi - C_\psi$ is* (*weakly*) *compact on $\mathcal{B}^\alpha$ if and only if the following three conditions hold:*

$$\begin{gathered}
\mathcal{D}^\alpha \varphi(z)\, \rho\big(\varphi(z), \psi(z)\big) \to 0 \quad \text{as } |\varphi(z)| \to 1, \\
\mathcal{D}^\alpha \psi(z)\, \rho\big(\varphi(z), \psi(z)\big) \to 0 \quad \text{as } |\psi(z)| \to 1, \\
\mathcal{D}^\alpha \varphi(z) - \mathcal{D}^\alpha \psi(z) \to 0 \quad \text{as } \min\big(|\varphi(z)|, |\psi(z)|\big) \to 1.
\end{gathered}$$

The first two conditions here are obvious analogues of those in Theorem 21. The third condition, however, turns out to be strictly necessary for the compactness of the difference operator. In Section 4 of [C] we actually construct symbols φ and ψ, both satisfying Madigan's boundedness condition, such that (8) holds but $C_\varphi - C_\psi$ is non-compact on $\mathcal{B}^\alpha$. In another direction, our work raises the open question whether condition (8) (albeit not being sufficient) could be necessary for the compactness of $C_\varphi - C_\psi$ on $\mathcal{B}^\alpha$ with $0 < \alpha < 1$, as it is in the larger space H^∞.

We approach the proofs of Theorems 21 and 22 from a rather general standpoint where we are concerned with differences of weighted composition operators on weighted H^∞-type spaces. This is motivated by the papers of Contreras and Hernández-Díaz [7] and Montes-Rodríguez [20] dealing with the case of a single operator. More specifically, we let $\alpha > 0$ and consider the space H^∞_α of analytic functions $f \colon \mathbb{D} \to \mathbb{C}$ such that

$$\sup_{z \in \mathbb{D}} (1 - |z|^2)^\alpha |f(z)| < \infty.$$

Then H^∞_α is a Banach space under the norm given by this supremum. Also, if $\varphi \colon \mathbb{D} \to \mathbb{D}$ and $u \colon \mathbb{D} \to \mathbb{C}$ are analytic maps, we consider the weighted composition operator $W_{\varphi,u}$ taking an analytic function f onto $(f \circ \varphi) \cdot u$. Since $(C_\varphi f)' = (f' \circ \varphi) \cdot \varphi'$, it follows readily that the composition operator C_φ acting on $\mathcal{B}^\alpha$ is similar to the weighted composition operator $W_{\varphi,\varphi'}$ acting on H^∞_α, provided that in $\mathcal{B}^\alpha$ one identifies functions that differ by a constant.

Thus, in Section 2 of [C] we derive a general characterization for the compactness and weak compactness of an operator of the form $W_{\varphi,u} - W_{\psi,v}$ acting between two weighted H^∞ spaces. This stage of our work is in part parallel to a recent paper by Bonet, Lindström and Wolf [3], where the differences of (unweighted) composition operators between more general weighted spaces are studied. An application of the similarity argument described above then yields Theorem 22. To establish Theorem 21 we also make use of the continuity properties of hyperbolic derivatives to show that condition (9) is superfluous in that case.

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